· limit of a multi-variable functions $\varepsilon-\delta$.

· Proving limit does not exist by showing that two limits along different paths are different.

· squeeze thm

Finding limit using polar coordinates.

Recall $(x,y) \leftrightarrow (r,\theta)$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

In particular, $(x,y) \to (0,0) \iff r \to 0$

Example ① $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

$$= \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2}$$

$$= \lim_{r\to 0} r\cos^3\theta + r\sin^3\theta$$

$$= 0$$

$|\cos^3\theta + \sin^3\theta|$

$\leq |\cos^3\theta| + |\sin^3\theta|$

≤ 2

$\therefore |r(\cos^3\theta + \sin^3\theta)|$

$\leq 2|r|$

Also, $\lim\limits_{r\to 0} 2|r| = 0$

$\therefore$ By the squeeze theorem,

$\lim\limits_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$.

② $$\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)}$$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta + r^2\cos\theta\sin\theta}{2(r^2)}$$

$$= \lim_{r\to 0} \frac{\cos^2\theta + \cos\theta\sin\theta}{2}$$

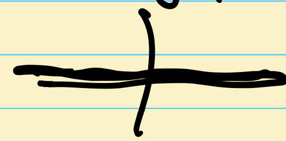
$$= \frac{\cos^2\theta + \cos\theta\sin\theta}{2}$$

this value does depend on θ.

if $\theta = 0$, $\frac{1}{2}$

if $\theta = \frac{\pi}{2}$, 0.

(→ Taking limit along path x-axis



→ along path y-axis)

∴ limit does not exist.

③ $$\lim_{(x,y)\to(0,0)} xy\ln(x^2+y^2) = \lim_{r\to 0} r^2\cos\theta\sin\theta\ln(r^2)$$

(as $r\to 0$
$r^2 \to 0$
$\ln(r^2) \to -\infty$)

$|\cos\theta\sin\theta| \le 1$

hence $|r^2\cos\theta\sin\theta\ln(r^2)| \le |r^2\ln(r^2)|$

$$\lim_{r\to 0} r^2\ln(r^2) = \lim_{r\to 0}\frac{\ln(r^2)}{\frac{1}{r^2}} = \lim_{r\to 0}\frac{\frac{2r}{r^2}}{-2\cdot\frac{1}{r^3}}$$

$$= \lim_{r\to 0} -r^2 = 0.$$

$\therefore$ By squeeze theorem,

$$\lim_{(x,y)\to(0,0)} xy\ln(x^2+y^2) = 0$$

Iterated limit

$f : \mathbb{R}^2 \to \mathbb{R}$

$\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ = taking limit w.r.t. $y\to 0$ and then taking limit w.r.t. $x \to 0$.

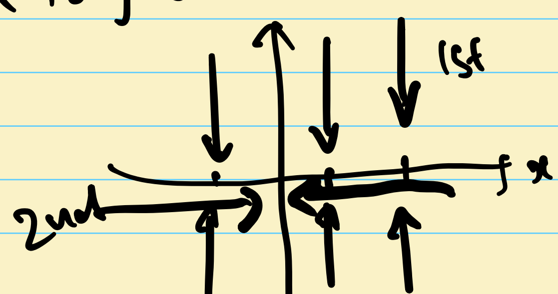
Similarly, we can consider $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$.

<u>Q</u> Are they equal?

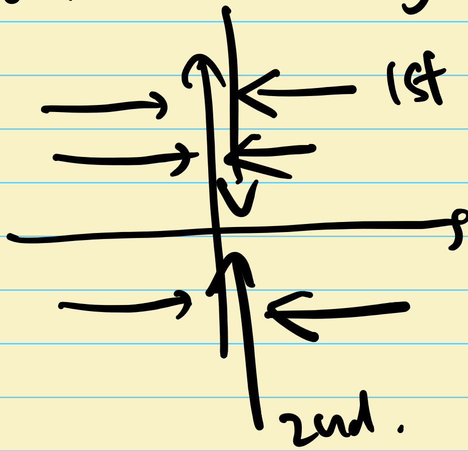
$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \quad \text{vs} \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y) \quad \text{vs} \quad \lim_{(x,y)\to(0,0)} f(x,y)$$

<u>Example</u> $f(x,y) = \dfrac{x+y}{x-y}$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\frac{x+0}{x-0} = \lim_{x\to 0} 1 = 1.$$



$$\lim_{y\to0}\lim_{x\to0} f(x,y) = \lim_{y\to0}\frac{y}{-y} = \lim_{y\to0} -1 = -1.$$



$$\lim_{(x,y)\to(0,0)}\frac{x+y}{x-y} = \begin{cases} \text{along } x=-y\ ;\ 0 \\ \text{along } y=0\ ;\ 1 \end{cases}$$ not equal

$\therefore$ " does not exist.

<u>Rmk</u> ① $\lim_{x\to0}\lim_{y\to0} f(x,y) = \lim_{y\to0}\lim_{x\to0} f(x,y)$ both exists and equal $\not\Rightarrow$ $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

e.g. $f(x,y) = \begin{cases} 1 & x=y \\ 0 & x\neq y \end{cases}$

$\not\Leftarrow$

e.g. $f(x,y) = \begin{cases} x\cos\frac{1}{y} + y\cos\frac{1}{x} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

(exercise: check this)

② If all the three limits exist, they are equal.

Continuity Let $f : A (\subseteq \mathbb{R}^n) \to \mathbb{R}$, $\vec{a} \in A$.

Def f is continuous at $\vec{a} \in A$ if

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

if $\vec{x} \in A$ and $\|\vec{x} - \vec{a}\| < \delta$, then

$$|f(\vec{x}) - f(\vec{a})| < \varepsilon.$$

Equivalent definition f is continuous at $\vec{a}$ if $\lim_{\vec{x} \to \vec{a}} f(\vec{x})$ exists and equal to $f(\vec{a})$.

Def f is continuous if f is continuous at any point in A.

eg $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, \cdots x_n) = x_k$ $\quad 1 \le k \le n$

f is continuous.

(pf) We need to check f is continuous at any $\vec{a} = (a_1, \cdots a_n) \in \mathbb{R}^n$.

We need to check that $\forall \varepsilon > 0$, we can find a $\delta > 0$ s.t. $\|\vec{x} - \vec{a}\| < \delta \Rightarrow |f(\vec{x}) - f(\vec{a})| < \varepsilon$.

Pick $\delta = \varepsilon$, then

$$\|\vec{x} - \vec{a}\| = \sqrt{(x_1-a_1)^2 + \cdots + (x_n-a_n)^2}$$

$$|f(\vec{x}) - f(\vec{a})| = |x_k - a_k|$$

$$\underset{\sqrt{(x_k-a_k)^2}}{\overset{}{|x_k - a_k|}} \le \sqrt{(x_1-a_1)^2 + \cdots + (x_n-a_n)^2} < \delta = \varepsilon.$$

$\therefore f$ is continuous at $\vec{a}$.

This is true for any $\vec{a}$, $\therefore f$ is continuous.

<u>Thm</u> If $f, g : \Omega\ (\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}$ are continuous at $\vec{a}$, then

① $f(\vec{x}) \pm g(\vec{x})$, $kf(\vec{x})$, $f(\vec{x})g(\vec{x})$ are continuous at $\vec{a}$. $(k \in \mathbb{R})$

② If $g(\vec{a}) \neq 0$, then $\dfrac{f(\vec{x})}{g(\vec{x})}$ is continuous at $\vec{a}$.

(proof) Follows from corresponding properties of limits. $\square$

eg All polynomials are continuous.

eg $f(x,y,z): \mathbb{R}^3 \to \mathbb{R}$

$$= x^3 + 3yz + z^2$$

We proved in the above example that

x, y, z are continuous

$\overset{\Rightarrow}{\text{Thm}}$ $x^3 = x \cdot x \cdot x$

$3yz = 3 \cdot y \cdot z$ are all continuous

$z^2 = z \cdot z$

$\overset{\Rightarrow}{\text{Thm}}$ $x^3 + 3yz + z^2$ is continuous.

Also, rational functions $\left(\frac{\text{poly}}{\text{poly}}\right)$ are continuous by ② of Thm.

e.g. $\dfrac{x^3+y^3+yz}{x^2+y^2}$ is continuous on

$\mathbb{R}^3 \setminus \{(x,y,z) \mid x^2+y^2=0\}$

$= \mathbb{R}^3 \setminus \{z\text{-axis}\}$

Let $Q(\vec{x}) = \dfrac{p_1(\vec{x})}{p_2(\vec{x})}$ be a rational function.

Suppose $p_2(\vec{a}) = 0$. $\rightsquigarrow$ $Q(\vec{x})$ is not defined at $\vec{a}$.

$Q(\vec{x})$ can be extended to a function continuous at $\vec{a}$ $\Leftrightarrow$ $\lim_{\vec{x}\to\vec{a}} Q(\vec{x})$ exists.

eg ① $Q(x,y) = \dfrac{x^2-y^2}{x-y}$

$Q(x,y)$ is not defined on $\{x=y\}$.

$\dfrac{x^2-y^2}{x-y} = x+y$ is defined on whole $\mathbb{R}^2$.

for any $(a,a) \in \{x=y\}$,

$\lim_{(x,y)\to(a,a)} Q(x,y) = 2a$ exists.

② $f(x,y) = \dfrac{xy+y^3}{x^2+y^2}$. f is not defined at $(0,0)$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} \frac{mx^2+m^3x^3}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{m+m^3x}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

$\frac{m}{1+m^2}$ depend on m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore$ f cannot be extended to a continuous function on $\mathbb{R}^2$.

③ $g(x,y) = \frac{x^4-y^4}{x^2+y^2}$ is not defined at $(0,0)$.

Can g be extend to a cont function on $\mathbb{R}^2$?

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{r\to 0} \frac{r^4(\cos^4\theta - \sin^4\theta)}{r^2}$$

$$= \lim_{r\to 0} r^2(\cos^4\theta - \sin^4\theta)$$

$$= 0$$

$|r^2(\cos^4\theta - \sin^4\theta)| \le 2r^2$

$\lim_{r\to 0} 2r^2 = 0$

By squeeze theorem, limit is 0.

$$\tilde{g}(x,y) = \begin{cases} \frac{x^4-y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(or it is just x^2+y^2)

Thm If $f : \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}$ is continuous at $\vec{a}$. and g is 1-variable function continuous at $f(\vec{a})$. Then $g \circ f$ is continuous at $\vec{a}$.

$$\lim_{\vec{x} \to \vec{a}} g(f(\vec{x})) = g\left(\lim_{\vec{x} \to \vec{a}} f(\vec{x})\right)$$

$$= g(f(\vec{a}))$$

eg Let $g(x) = |x|$.

$f(x_1 - x_n) = x_k$.

Both f and g are continuous.

$\therefore$ By Thm, $g \circ f(x_1 \cdots x_n) = |x_k|$ is continuous.

eg $\sin(x^2 + yz)$, e^{x-y}, $\cos\left(\frac{1}{x^2+y^2}\right)$, $\sqrt{x^2+y^2}$

are continuous on their domain

Partial derivatives

: rate of change of a function with respect to each variable.

Def $f : \Omega (\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}$, Ω open

For $i = 1, \cdots, n$ define i-th partial derivative of f at $\vec{x} = (x_1 \cdots, x_n) \in \Omega$ to be

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \lim_{h \to 0} \frac{f(x_1 \cdots x_i + h \cdots x_n) - f(x_1 \cdots x_n)}{h}$$

eg In $n = 2$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

<u>notations</u>

$$\frac{\partial f}{\partial x} = \partial_1 f = D_1 f = f_x$$

$$\frac{\partial f}{\partial y} = \partial_2 f = D_2 f = f_y.$$

eg $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{((x+h)^2 + y^2) - (x^2 + y^2)}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

(Taking derivative regarding y as constant)

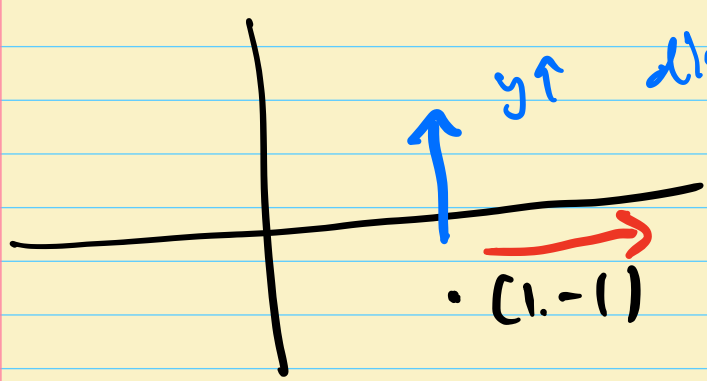
$$\frac{\partial f}{\partial y} = 2y$$

$\frac{\partial f}{\partial x}(1,-1) = 2 > 0 \Rightarrow f$ increases as x increases at $(1,-1)$

$\frac{\partial f}{\partial y}(1,-1) = -2 < 0 \Rightarrow f$ decreases as y increases at $(1,-1)$

Rmk $f(x,y) = \|(x,y)\|^2$

$y\uparrow$ distance from $\vec{0}$ $\downarrow$, so $f\downarrow$



$x\uparrow$, distance from $\vec{0}$ $\uparrow$
so $f\uparrow$

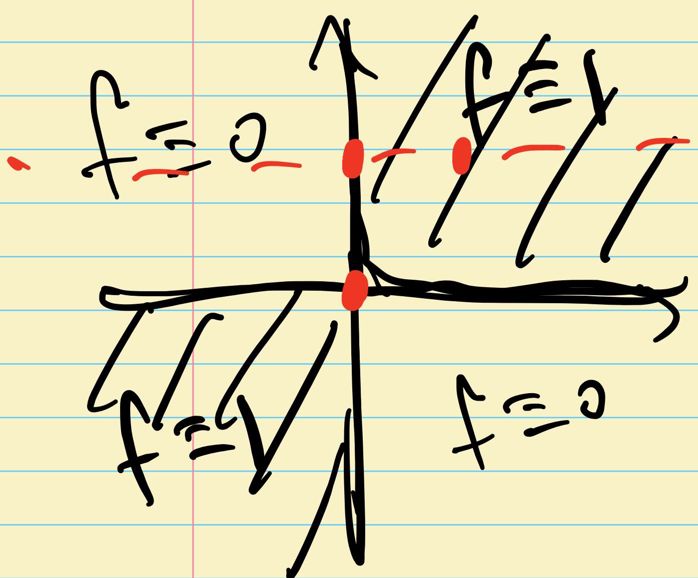
eg $f(x,y,z) = xy^2 - \cos(xz)$

Then $f_x = y^2 + z\sin(xz)$

$f_y = 2xy$

$f_z = x\sin(xz)$.

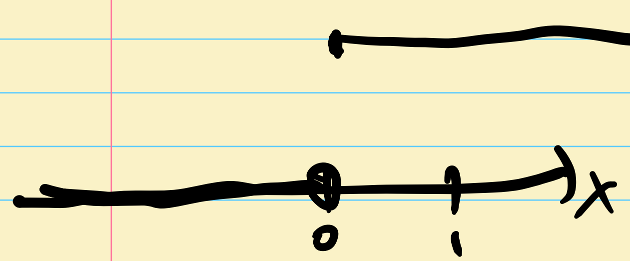
eg $f(x,y) = \begin{cases} 1 & \text{if } xy \geq 0 \\ 0 & \text{if } xy < 0 \end{cases}$



What are $\frac{\partial f}{\partial x}(1,1)$, $\frac{\partial f}{\partial x}(0,1)$, $\frac{\partial f}{\partial x}(0,0)$?

$\frac{\partial f}{\partial x}$ is fixing y, differentiate w.r.t. x.

Along $y=1$ $(\cdots)$ $f(x,1) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$



$\therefore \frac{\partial f}{\partial x}(1,1) = 0$

$\frac{\partial f}{\partial x}(0,1)$ does not exist

At $(0,0)$, along $y=0$.

$f(x,0) \equiv 0$ for any $x \in \mathbb{R}$.

$\therefore \dfrac{\partial f}{\partial x}(0,0) = 0$.

Rmk $\dfrac{\partial f}{\partial x}(0,0) = 0$ exist.

Note that f is not continuous at $(0,0)$

$\therefore \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial x}$ exist at $(0,0)$

$\not\Rightarrow$ f is continuous at $(0,0)$.